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Twisting type N vacuums with cosmological constant

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Dedicated to the memory of Jerzy F. Plebański

Abstract

We provide the first examples of vacuum metrics with cosmological constant which have a twisting quadruple principal null direction.

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In a joint paper with Plebański [10] we provided, among other things, a maximally reduced system of vacuum Einstein equations

$$\mathrm{Ric}(g) = \Lambda g$$

for *twisting type N spacetimes* (see [13], Ch. 29, for a definition). After writing down the reduced equations we concluded the first section of [10] with a remark that it is very difficult to find a single solution to the presented equations when $\Lambda \neq 0$.

Actually the problem of finding twisting type N vacuums, with the *cosmological constant* $\Lambda = 0$ or not, is one of the hardest in the theory of algebraically special solutions. If $\Lambda = 0$, the only *explicit* solution is that of Hauser [3]. It is *explicit* in the sense that it can be expressed in terms of hypergeometric functions. Very few other twisting type N vacuums with $\Lambda = 0$ are known. Unlike Hauser's solution they are not that explicit any longer. At best they may be expressed in terms of a *quite complicated nonlinear ODE of the third order* [1,2,4,8].

The main difficulty with twisting type N vacuums with $\Lambda = 0$ is that the equations are strongly overdetermined, and it is very difficult to guess an ansatz which will not lead to the Minkowski metric as the only solution.

Having all this in mind we present the following metric:

$$g = \frac{1}{s^2 y^2 \cos^2(\frac{r}{2})} \times \left(\frac{3}{2}(\mathrm{d}x^2 + \mathrm{d}y^2) + (\mathrm{d}x + y^3 \mathrm{d}u)\left(y\mathrm{d}r + \frac{1}{3}y^3 \cos r\,\mathrm{d}u + \left(2 + \frac{7}{3}\cos r\right)\mathrm{d}x + 2\sin r\,\mathrm{d}y\right)\right). \tag{0.1}$$

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Here (x, y, u, r), with ranges $-\infty < x, u < \infty$, $0 < |y| < \infty$, $|r| < \pi$, are coordinates of the spacetime described by g; the quantity s is a *real* nonzero constant, which we regard as a free parameter.

A short calculation[1] leads to the following remarkable

Proposition 0.1. *The metric (0.1) satisfies the* vacuum Einstein′s equations

$$\mathrm{Ric}(g) = -s^2 g$$

with a negative *cosmological constant* $\Lambda = -s^2$.

The metric is of Petrov type N . *In an appropriate null coframe its* only nonvanishing *Weyl spinor coefficient is*

$$\Psi_4 = \frac{14}{3}\frac{s^2}{y^2}\mathrm{e}^{-\frac{\mathrm{i}r}{2}}\cos^3\frac{r}{2}, \quad \textit{with } \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0.$$

The vector field $k = \partial_r$ *is tangent to a* twisting *congruence of null and shear-free geodesics, which is* aligned *with the* quadruple principal null direction *of the Weyl tensor for* g.

Thus the metric (0.1) provides an example of a *twisting type N Einstein metric with* negative *cosmological constant*; an example that we were missing when writing the joint paper [10] with Plebański.

The following remarks are in order:

Remark 0.2. The twisting congruence of shear-free and null geodesics tangent to the vector field $k = \partial_r$ of Proposition 0.1 corresponds to a quite nontrivial three-dimensional CR structure [5]. This is *not* CR equivalent, even locally, to the Heisenberg group CR structure [5], which is the underlying CR structure of the Hauser solution [12]. This implies that the solution (0.1) does not lead to a Hauser metric by means of any kind of limiting procedure, such as $\Lambda \to 0$. It is quite ironic that it is the same CR structure as was used in [10], Section 5, to construct an example of a metric which satisfies the Bach equations and which is not conformal to an Einstein metric.

Remark 0.3. The metric (0.1) is not accidental. It appears as one of the simplest examples in our recent formulation of the twisting vacuum Einstein equations with cosmological constant [5]. In [5] we generalized our earlier results [7,9] on four-dimensional Lorentzian spacetimes $(\mathcal{M}, g)$ satisfying the Einstein equations $R_{\mu\nu} = \Phi k_\mu k_\nu$, with k_μ being a vector field tangent to a *twisting* congruence of *null* and *shear-free geodesics*, to the case of a *nonvanishing* cosmological constant Λ. We proved that the metric g of a spacetime $(\mathcal{M}, g)$ satisfying

$$R_{\mu\nu} = \Lambda g_{\mu\nu} + \Phi k_\mu k_\nu, \tag{0.2}$$

with k as above, *factorizes* as

$$g = \Omega^{-2}\hat{g}, \qquad \Omega = \cos\left(\frac{r}{2}\right), \tag{0.3}$$

where $\hat{g}$ is *periodic* in terms of the null coordinate r along $k = \partial_r$. We further showed that if g satisfies (0.2), then $\mathcal{M}$ is a *circle bundle* $\mathbb{S}^1 \to \mathcal{M} \to M$ over a three-dimensional strictly pseudoconvex CR manifold $(M, [(\lambda, \mu)])$, and that

$$\hat{g} = p^2[\mu\bar{\mu} + \lambda(\mathrm{d}r + W\mu + \bar{W}\bar{\mu} + H\lambda)], \tag{0.4}$$

with

$$W = \mathrm{i}a\mathrm{e}^{-\mathrm{i}r} + b, \qquad H = \frac{n}{p^4}\mathrm{e}^{2\mathrm{i}r} + \frac{\bar{n}}{p^4}\mathrm{e}^{-2\mathrm{i}r} + q\mathrm{e}^{\mathrm{i}r} + \bar{q}\mathrm{e}^{-\mathrm{i}r} + h. \tag{0.5}$$

[1] Nowadays this can be done by a symbolic computer calculation package!

Here λ (real) and μ (complex) are 1-forms on $\mathcal{M}$ such that $k \lrcorner \lambda = k \lrcorner \mu = 0$, $k \lrcorner \mathrm{d}\lambda = k \lrcorner \mathrm{d}\mu = 0$, $\lambda \wedge \mu \wedge \bar{\mu} \neq 0$ and the functions a, b, n, q (complex) and p, h (real), all of which are independent of r, satisfy

$$\begin{aligned}
a &= c + 2\partial \log p \\
b &= \mathrm{i}c + 2\mathrm{i}\partial \log p \\
q &= \frac{3n+\bar{n}}{p^4} + \frac{2}{3}\Lambda p^2 + \frac{2\partial p \bar{\partial} p - p(\partial\bar{\partial} p + \bar{\partial}\partial p)}{2p^2} - \frac{\mathrm{i}}{2}\partial_0 \log p - \bar{\partial} c \\
h &= 3\frac{n+\bar{n}}{p^4} + 2\Lambda p^2 + \frac{2\partial p \bar{\partial} p - p(\partial\bar{\partial} p + \bar{\partial}\partial p)}{p^2} - \bar{\partial} c - \partial \bar{c}.
\end{aligned} \tag{0.6}$$

Here the r-independent complex function c is defined via

$$\begin{aligned}
&\mathrm{d}\mu = 0, \qquad \mathrm{d}\bar{\mu} = 0, \\
&\mathrm{d}\lambda = \mathrm{i}\mu \wedge \bar{\mu} + (c\mu + \bar{c}\bar{\mu}) \wedge \lambda,
\end{aligned} \tag{0.7}$$

and the operators $(\partial_0, \partial, \bar{\partial})$ are vector fields on M, which are algebraic dual to the coframe $(\lambda, \mu, \bar{\mu})$ on the CR manifold M. The only unknowns, n and p, satisfy the following system of PDEs on M:

$$\partial n + 3cn = 0, \tag{0.8}$$

$$\left[\partial\bar{\partial} + \bar{\partial}\partial + \bar{c}\partial + c\bar{\partial} + \frac{1}{2}c\bar{c} + \frac{3}{4}(\partial\bar{c} + \bar{\partial}c)\right] p = \frac{n+\bar{n}}{p^3} + \frac{2}{3}\Lambda p^3. \tag{0.9}$$

Note that the above functions a, b, c, h, n, p, q define a *vacuum* metric g via (0.3) iff, in addition to Eqs. (0.8) and (0.9), the functions n and p satisfy also the equation $\Phi = 0$. This is a quite nasty nonlinear PDE relating n, p and c.

In this context our solution (0.1) is very simple. It is given by

$$\begin{aligned}
&\lambda = \frac{2y^4}{3}\left(\mathrm{d}u + \frac{\mathrm{d}x}{y^3}\right), \qquad \mu = \mathrm{d}x + \mathrm{i}\mathrm{d}y, \qquad c = -\frac{2\mathrm{i}}{y}, \\
&n = 0, \qquad p = \frac{\sqrt{3}}{2sy}, \qquad s^2 = -\Lambda,
\end{aligned}$$

as can be easily checked by a direct calculation using definitions (0.3)–(0.7). It is a miracle that in *addition* to the pure radiation Einstein equations (0.8) *and* (0.9), this solution satisfies the *vacuum* condition $\Phi = 0$, and simultaneously the *type N* conditions $\Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0$, $\Psi_4 \neq 0$. This shows that the formulation of the twisting Einstein equations described in [5,7,9] has an unexplored power.

Remark 0.4. Actually the solution (0.1) is only the simplest one from a larger class of twisting type N vacuums with cosmological constant contained in (0.3)–(0.9). These solutions are obtained by taking

$$\begin{aligned}
&\lambda = -\frac{2}{f'(y)}(\mathrm{d}u + f(y)\mathrm{d}x), \qquad \mu = \mathrm{d}x + \mathrm{i}\mathrm{d}y, \qquad c = \frac{\mathrm{i}f''(y)}{2f'(y)}, \\
&n = 0, \qquad p = p(y),
\end{aligned} \tag{0.10}$$

with real functions $f = f(y)$ and $p = p(y)$. With this choice the twisting type N vacuum Einstein equations with cosmological constant are equivalent to the following system of two third-order ODEs:

$$\begin{aligned}
&\frac{9}{4}pf'f''' + 3p'f'f'' - 3pf''^2 - 3p''f'^2 + 4\Lambda p^3 f'^2 = 0, \\
&3pf'\left(3f'^2p''' + 2f''(9p'f'' - 5f'p'')\right) + 3f'^2p'(5f'p'' - 14p'f'') - 12p^2f''^3 \\
&\quad = 152\Lambda p^3 f'^2(pf'' - 2p'f').
\end{aligned} \tag{0.11}$$

It can be checked by direct calculation that every solution $p = p(y)$, $f = f(y)$ of these two equations, after being inserted into (0.10) and (0.3)–(0.6), gives a metric g, which satisfies Einstein's vacuum equations $\mathrm{Ric}(g) = \Lambda g$ and is of Petrov type N, having $k = \partial_r$ tangent to a twisting congruence of null geodesics. The only nonvanishing Weyl spin coefficient for these metrics is

$$\Psi_4 = -\frac{2e^{-\frac{ir}{2}}\cos^3\frac{r}{2}}{27p^2 f'^2}\Lambda\left(8\Lambda p^4 f'^2 + 27f'^2 p'^2 + 12p^2 f''^2 + 3pf'(f'p'' - 13p'f'')\right).$$

Eq. (0.11) admit nontrivial solutions. The simplest of them leads to our metric (0.1). Note that if $\Lambda \to 0$, then also the Weyl coefficient $\Psi_4 \to 0$. Hence the metric g becomes flat. This shows that none of the solutions (0.11) can be obtained as a Λ-deformation of the Hauser solution.

Remark 0.5. Since, as discussed in [6], every metric (0.3)–(0.9) is relevant for Penrose's 'before the big bang' argument [11], the type N metrics corresponding to solutions of (0.11) constitute a nice set of explicit examples in which Penrose's ideas can be tested.

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